

STABLE SHORT EXACT SEQUENCES DEFINE AN EXACT STRUCTURE ON ANY ADDITIVE CATEGORY

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ABSTRACT. Rump has recently showed the existence of a unique maximal Quillen exact structure on any additive category. We prove that this is given precisely by the stable short exact sequences, i.e. kernel-cokernel pairs consisting of a semi-stable kernel and a semi-stable cokernel.

1. INTRODUCTION

Exact categories provide a suitable setting for developing homological algebra beyond abelian categories, and have important applications in different fields, such as algebraic K -theory, algebraic geometry and topology, algebraic and functional analysis etc., where many relevant categories have a poorer algebraic structure than an abelian category. The most important concept of exact category in the additive case has crystalized in the work of Quillen on algebraic K -theory [13], and has been simplified by Keller [9]. This extends different previous notions such as those of Heller [7] and Yoneda [21], to mention only the closest ones. A Quillen exact category is an additive category endowed with a distinguished class of kernel-cokernel pairs, called *conflations*, satisfying certain axioms (see Section 4). Several recent papers present exhaustive accounts on Quillen exact categories [2, 4], and rebring into the main attention their power, visible in some new applications, such as those in functional analysis [4, 19], model structures [5] or approximation theory [17].

Every additive category has a smallest exact structure, whose conflations are the split exact sequences. It is natural to wonder if there exists, and then which is, the greatest exact structure on an arbitrary additive category. More important than its maximality it is the possibility to have a natural exact structure in any additive category, apart from the trivial one given by the split exact sequences, with respect to which to develop homological algebra. Since every conflation is a kernel-cokernel pair, a first candidate for the class of conflations should obviously be the maximal possible class in the definition of an exact category, namely that of all kernel-cokernel pairs. While this is the right choice for abelian categories, or even for quasi-abelian categories (i.e. additive categories with kernels and cokernels such that pushouts preserve kernels and pullbacks preserve cokernels) [6, 15, 18], it is no longer suitable for the more general setting of preabelian categories (i.e. additive categories with kernels and cokernels).

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It has already been observed by Richman and Walker [14, p. 522] that the class of all kernels (cokernels) in a preabelian category is not closed under pushouts (pullbacks), and so the class of all kernel-cokernel pairs cannot define an exact structure in this setting. In order to study homological algebra in preabelian categories they introduced *semi-stable* kernels (cokernels) as those kernels (cokernels) which are preserved by pushouts (pullbacks), and the *stable* short exact sequences as those kernel-cokernel pairs consisting of a semi-stable kernel and a semi-stable cokernel. Recently, Sieg and Wegner have made use of these concepts, and showed that the class of stable short exact sequences defines the unique maximal exact structure on any preabelian category [19, Theorem 3.3]. A natural extension of the definition of semi-stable kernels and semi-stable cokernels from a preabelian category to an arbitrary additive category allowed the generalization of the above result in [3, Theorem 3.5], which shows that the class of stable short exact sequences defines the unique maximal exact structure on any weakly idempotent complete additive category. Note that many classes of additive categories, such as the accessible categories (which have a natural exact structure consisting of the pure exact sequences) [12] and the triangulated categories (which only have the trivial exact structure), are weakly idempotent complete. But there are additive categories (even exact) which are not weakly idempotent complete, for instance any category of free modules in which there exist projective modules which are not free (e.g., see [5, p. 2894]).

The remaining challenge was to determine a greatest exact structure on any additive category. A step towards that direction has recently been made by Rump, which shows that there exists a greatest exact structure on any additive category [16], without effectively determining it. His interesting approach uses a new concept of one-sided exact category (also, see [1]), by constructing the maximal left exact structure and the maximal right exact structure, and then deducing the existence of the greatest exact structure. Nevertheless, the question whether this is defined by the class of stable kernel-cokernel pairs has still remained open. The present paper answers this in the affirmative, and completes the problem of determining the greatest exact structure on any additive category. The main obstacle so far has been to prove that the semi-stable kernels and the semi-stable cokernels satisfy Quillen's "obscure axiom" (see Proposition 3.3 and its dual). Our approach uses essentially the property that every additive category has an additive idempotent completion (or Karoubian completion) [8, p. 75], which in turn is weakly idempotent complete [2]. The technique is to establish some relative versions of our previous results on semi-stable kernels and semi-stable cokernels in (weakly idempotent complete) additive categories given in [3], and use the canonical fully faithful functor $H : \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ between an additive category \mathcal{C} and its idempotent completion $\widehat{\mathcal{C}}$ in order to transfer properties back and forth. We apply our theorem to categories of chain complexes and categories of projective spectra.

2. STABLE SHORT EXACT SEQUENCES

The notion of stable short exact sequence was introduced in [14] for preabelian categories, and generalized to arbitrary additive categories in [3]. We shall need the following relative version of this concept.

Definition 2.1. Let \mathcal{C} be an additive category and let \mathcal{H} be a class of objects in \mathcal{C} containing the zero object and closed under isomorphisms.

A cokernel $d : B \rightarrow C$ is called an \mathcal{H} -semi-stable cokernel if there exists the pullback $(B' = B \times_C C', g, d')$ of d along any morphism $h : C' \rightarrow C$ with $C' \in \mathcal{H}$, and d' is again a cokernel, i.e. there exists a pullback square

$$\begin{array}{ccc} B' & \xrightarrow{d'} & C' \\ g \downarrow & & \downarrow h \\ B & \xrightarrow{d} & C \end{array}$$

with the morphism $d' : B' \rightarrow C'$ a cokernel. The notion of \mathcal{H} -semi-stable kernel is defined dually.

A short exact sequence, i.e. a kernel-cokernel pair, $A \xrightarrow{i} B \xrightarrow{d} C$ is called \mathcal{H} -stable if i is an \mathcal{H} -semi-stable kernel and d is an \mathcal{H} -semi-stable cokernel.

For $\mathcal{H} = \mathcal{C}$ the above concepts particularize to semi-stable kernels, semi-stable kernels and stable short exact sequences.

Let us note some useful remarks, which will be freely used together with their dual versions.

Remark 2.2. (i) Every \mathcal{H} -semi-stable cokernel $d : B \rightarrow C$ has a kernel (namely, its pullback along the morphism $0 \rightarrow C$; note that \mathcal{H} contains the zero object). Hence every \mathcal{H} -semi-stable cokernel is the cokernel of its kernel (e.g., by the dual of [20, Chapter IV, Proposition 2.4]).

(ii) The pullback of an \mathcal{H} -semi-stable cokernel along any morphism with the domain in \mathcal{H} exists and is again an \mathcal{H} -semi-stable cokernel.

(iii) Every isomorphism is an \mathcal{H} -semi-stable cokernel.

(iv) For every object B of \mathcal{C} , the morphism $B \rightarrow 0$ is an \mathcal{H} -semi-stable cokernel.

Throughout the paper \mathcal{C} will be an *additive* category, and \mathcal{H} will be a *non-empty class of objects in \mathcal{C} closed under pullbacks and pushouts*, in the sense that if $d : B \rightarrow C$ and $h : C' \rightarrow C$ are morphisms with $B, C' \in \mathcal{H}$ having a pullback, then $B \times_C C' \in \mathcal{H}$, and its dual statement. In particular, \mathcal{H} is closed under kernels, cokernels and finite biproducts (denoted by \oplus , and referred to as direct sums), and moreover, \mathcal{H} fits into Definition 2.1.

We shall need the following two well-known results on pullbacks in an arbitrary category \mathcal{C} , whose duals for pushouts hold as well.

Lemma 2.3. [10, Lemma 5.1] *Consider the following diagram in \mathcal{C} in which the squares are commutative and the right square is a pullback:*

$$\begin{array}{ccccc} A' & \xrightarrow{i'} & B' & \xrightarrow{d'} & C' \\ f \downarrow & & g \downarrow & & \downarrow h \\ A & \xrightarrow{i} & B & \xrightarrow{d} & C \end{array}$$

Then the left square is a pullback if and only if so is the rectangle.

Lemma 2.4. [14, Theorem 5] *Let $d : B \rightarrow C$ and $h : C' \rightarrow C$ be morphisms in \mathcal{C} such that d has a kernel $i : A \rightarrow B$, and the pullback of d and h exists. Then there exists a commutative diagram in \mathcal{C} :*

$$\begin{array}{ccccc} A & \xrightarrow{i'} & B' & \xrightarrow{d'} & C' \\ \parallel & & \downarrow g & & \downarrow h \\ A & \xrightarrow{i} & B & \xrightarrow{d} & C \end{array}$$

in which the right square is a pullback and $i' : A \rightarrow B'$ is the kernel of d' .

We shall prove some essential properties of \mathcal{H} -semi-stable cokernels. Clearly, their dual versions for \mathcal{H} -semi-stable kernels hold as well. These results on relative semi-stable cokernels are modelled after the corresponding ones for semi-stable cokernels from [3]. We sketch the proofs and point out where the class \mathcal{H} intervenes.

Proposition 2.5. *The composition of two \mathcal{H} -semi-stable cokernels $d : B \rightarrow C$ and $p : C \rightarrow D$ in \mathcal{C} with $C \in \mathcal{H}$ is an \mathcal{H} -semi-stable cokernel.*

Proof. Since d and p are \mathcal{H} -semi-stable cokernels, we have $d = \text{coker}(i)$ and $p = \text{coker}(h)$, where $i = \ker(d) : A \rightarrow B$ and $h = \ker(p) : C' \rightarrow C$. Since $C' \in \mathcal{H}$, there exists the pullback $(B' = B \times_C C', g, d')$ of d and h , and by Lemma 2.4 we have the following diagram with commutative squares:

$$\begin{array}{ccccc} A & \xrightarrow{i'} & B' & \xrightarrow{d'} & C' \\ \parallel & & \downarrow g & & \downarrow h \\ A & \xrightarrow{i} & B & \xrightarrow{d} & C \\ & & \downarrow pd & & \downarrow p \\ & & D & \xlongequal{\quad} & D \end{array}$$

As in the proof of [3, Proposition 3.1], it follows that $pd = \text{coker}(g)$.

In order to get a pullback of $pd : B \rightarrow D$ and an arbitrary morphism $\gamma : G \rightarrow D$ with $G \in \mathcal{H}$, first consider a pullback of p and γ , say $(C \times_D G, \beta, v)$. Since $C \times_D G \in \mathcal{H}$, there is a pullback of d and β , say $(B \times_C (C \times_D G), \alpha, u)$. Then $(B \times_D G, \alpha, vu)$ is a pullback of pd and γ by Lemma 2.3. Both u and v are \mathcal{H} -semi-stable cokernels, because so are d and p . Moreover, by the first part of the proof and the fact that $C \times_D G \in \mathcal{H}$, vu is a cokernel as the composition of the two \mathcal{H} -semi-stable cokernels v and u . Therefore, pd is an \mathcal{H} -semi-stable cokernel. \square

Proposition 2.6. *The direct sum of two \mathcal{H} -semi-stable cokernels $d : B \rightarrow C$ and $d' : B' \rightarrow C'$ in \mathcal{C} with $B', C \in \mathcal{H}$ is an \mathcal{H} -semi-stable cokernel.*

Proof. Consider the pullback square

$$\begin{array}{ccc} B \oplus B' & \xrightarrow{\begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix}} & C \oplus C' \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \downarrow & & \downarrow \begin{bmatrix} 1 & 0 \end{bmatrix} \\ B & \xrightarrow{d} & C \end{array}$$

Since $C \oplus B' \in \mathcal{H}$, $\begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix} : B \oplus B' \rightarrow C \oplus B'$ is an \mathcal{H} -semi-stable cokernel. Similarly, by the closure properties of \mathcal{H} , we have $C \oplus C' \in \mathcal{H}$ and $\begin{bmatrix} 1 & 0 \\ 0 & d' \end{bmatrix} : C \oplus B' \rightarrow C \oplus C'$ is an \mathcal{H} -semi-stable cokernel. Then their composition $\begin{bmatrix} d & 0 \\ 0 & d' \end{bmatrix} : B \oplus B' \rightarrow C \oplus C'$, that is $d \oplus d'$, is an \mathcal{H} -semi-stable cokernel by Proposition 2.5. \square

For the next proposition we need an extra assumption on our category. Recall that an additive category is called *weakly idempotent complete* if every retraction has a kernel (equivalently, every section has a cokernel) (e.g., see [2, Definition 7.2]).

Proposition 2.7. *Let \mathcal{C} be weakly idempotent complete. Let $d : B \rightarrow C$ and $p : C \rightarrow D$ be morphisms in \mathcal{C} with $B, C \in \mathcal{H}$ such that $pd : B \rightarrow D$ is an \mathcal{H} -semi-stable cokernel. Then p is an \mathcal{H} -semi-stable cokernel.*

Proof. Since pd is an \mathcal{H} -semi-stable cokernel and $C \in \mathcal{H}$, one may show as in the proof of [3, Proposition 3.4] that p has a kernel, say $h : C' \rightarrow C$. Let $g : B' \rightarrow B$ be the kernel of pd . Since pd is an \mathcal{H} -semi-stable cokernel, we have $pd = \text{coker}(g)$ and the following commutative left diagram:

$$\begin{array}{ccc} B' & \xrightarrow{d'} & C' \\ g \downarrow & & \downarrow h \\ B & \xrightarrow{d} & C \\ pd \downarrow & & \downarrow p \\ D & \xlongequal{\quad} & D \end{array} \qquad \begin{array}{ccc} B \oplus C' & \xrightarrow{[d \ h]} & C \\ [1 \ 0] \downarrow & & \downarrow p \\ B & \xrightarrow{pd} & D \end{array}$$

As in the proof of [3, Proposition 3.4], the right diagram is a pullback. Then $[d \ h]$ is a cokernel, because pd is an \mathcal{H} -semi-stable cokernel and $C \in \mathcal{H}$. Again as in the proof of [3, Proposition 3.4], we have $p = \text{coker}(h)$.

Now let $c : G \rightarrow D$ be a morphism with $G \in \mathcal{H}$. We shall show that there exists the pullback of p and c . We may write $[p \ 0]$ as the composition of the following morphisms:

$$C \oplus B \xrightarrow{\begin{bmatrix} 1 & -d \\ 0 & 1 \end{bmatrix}} C \oplus B \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & pd \end{bmatrix}} C \oplus D \xrightarrow{\begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix}} C \oplus D \xrightarrow{[0 \ 1]} D$$

The first and the third morphisms are isomorphisms, and so they are \mathcal{H} -semi-stable cokernels. The second morphism is an \mathcal{H} -semi-stable cokernel by Proposition 2.6, because $B, C \in \mathcal{H}$. Since $D \in \mathcal{H}$, Proposition 2.6 implies that the last morphism is an \mathcal{H} -semi-stable cokernel as the direct sum of the \mathcal{H} -semi-stable cokernels $C \rightarrow 0$ and 1_D . Therefore, the composition $[p \ 0]$ of the above four morphisms is also an \mathcal{H} -semi-stable cokernel by Proposition 2.5, because $C \oplus B, C \oplus D \in \mathcal{H}$. Hence $[p \ 0]$ and c have a pullback, say $(Y = (C \oplus B) \times_D G, \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix}, \gamma)$. Consider the morphism $\begin{bmatrix} 0 \\ 1 \end{bmatrix} : B \rightarrow C \oplus B$. Since $[p \ 0] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 = c0$, by the pullback property there is a unique morphism $\delta : B \rightarrow Y$ such that $\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} \delta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\gamma\delta = 0$. In particular, $\beta'\delta = 1_B$, and so β' is a retraction. Since \mathcal{C} is weakly idempotent complete, β' has a kernel, say $i : K \rightarrow Y$.

One shows that $(K, \alpha'i, \gamma i)$ is a pullback of p and c as in the proof of [3, Proposition 3.4]. Consider a pullback $(B \times_D G, \alpha'', \gamma')$ of pd and c . Then γ' factors through γi by the pullback property of $(K, \alpha'i, \gamma i)$. Since pd is an \mathcal{H} -semi-stable cokernel, so is γ' . Moreover, by Lemma 2.4 γi has a kernel, because so has p . Note that the two pullback properties imply that $K, B \times_D G \in \mathcal{H}$. Then γi must be a cokernel by an argument similar to the first part of the proof. Hence p is an \mathcal{H} -semi-stable cokernel. \square

3. IDEMPOTENT COMPLETION

An additive category is called *idempotent complete* if every idempotent morphism has a kernel [8, Définition 1.2.1]. A remarkable result states that every additive category has an *idempotent completion* (also called *Karoubian completion*) [8, Lemme 1.2.2]. More precisely, for every additive category \mathcal{C} , there exists an idempotent complete additive category $\widehat{\mathcal{C}}$ and a fully faithful additive functor $H : \mathcal{C} \rightarrow \widehat{\mathcal{C}}$. The category $\widehat{\mathcal{C}}$ has as objects the pairs (A, p) , where A is an object of \mathcal{C} and $p : A \rightarrow A$ is an idempotent morphism in \mathcal{C} , and as morphisms between two objects (A, p) and (B, q) of $\widehat{\mathcal{C}}$ the morphisms $f : A \rightarrow B$ in \mathcal{C} such that $f = qfp$. The biproduct in $\widehat{\mathcal{C}}$ is given by $(A, p) \oplus (B, q) = (A \oplus B, p \oplus q)$. The functor $H : \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ is defined by $H(A) = (A, 1_A)$ on objects A of \mathcal{C} , and by $H(f) = f$ on morphisms f in \mathcal{C} (also see [2, Section 6]). We denote by $\text{Im}(H)$ the essential image of H .

Lemma 3.1. *Let $(\widehat{\mathcal{C}}, H)$ be the idempotent completion of \mathcal{C} . Then $\text{Im}(H)$ is closed under pullbacks.*

Proof. We first prove that $\text{Im}(H)$ is closed under kernels. Let $j : X \rightarrow Y$ be a kernel in $\widehat{\mathcal{C}}$ with $Y \in \text{Im}(H)$. Then $X = (A, p)$ and $Y = H(B) = (B, 1_B)$ for some objects A, B and idempotent morphism $p : A \rightarrow A$ in \mathcal{C} . Hence $j : A \rightarrow B$ and $j = jp$ in \mathcal{C} . Let $r : B \rightarrow A$ be a left inverse of the monomorphism j . Then $p = rjp = rj = 1_A$, and so $X \in \text{Im}(H)$.

We claim that $\text{Im}(H)$ is also closed under finite direct sums. Indeed, if $X, X' \in \text{Im}(H)$, then $X = H(A)$ and $X' = H(A')$ for some objects A, A' of \mathcal{C} , and $X \oplus X' = (A, 1_A) \oplus (A', 1_{A'}) = (A \oplus A', 1_{A \oplus A'}) \in \text{Im}(H)$.

Finally, if $d : B \rightarrow C$ and $h : C' \rightarrow C$ are morphisms in $\widehat{\mathcal{C}}$ with $B, C' \in \text{Im}(H)$, then note that $(B' = B \times_C C', g, d')$ is the pullback of d and g if and only if $\begin{bmatrix} d' \\ -g \end{bmatrix} : B' \rightarrow C' \oplus B$ is the kernel of $\begin{bmatrix} h & d \end{bmatrix} : C' \oplus B \rightarrow C$. Now it follows that $\text{Im}(H)$ is closed under pullbacks. \square

Proposition 3.2. *Let $(\widehat{\mathcal{C}}, H)$ be the idempotent completion of \mathcal{C} . Then:*

- (i) *H preserves and reflects pullbacks.*
- (ii) *A morphism $d : B \rightarrow C$ in \mathcal{C} is a semi-stable cokernel if and only if $H(d)$ is an $\text{Im}(H)$ -semi-stable cokernel in $\widehat{\mathcal{C}}$.*

Proof. (i) Since H is fully faithful, it reflects pullbacks [11, Theorem 7.1].

Now let $(B \times_C C', g, d')$ be a pullback of some morphisms $d : B \rightarrow C$ and $h : C' \rightarrow C$ in \mathcal{C} . Let (D, p) be an object of $\widehat{\mathcal{C}}$ for some object D and idempotent morphism $p : D \rightarrow D$ in \mathcal{C} , and let $\alpha : (D, p) \rightarrow H(B)$, $\beta : (D, p) \rightarrow H(C')$ be morphisms in $\widehat{\mathcal{C}}$ such that $H(d)\alpha = H(h)\beta$. Then $\alpha : D \rightarrow B$, $\beta : D \rightarrow C'$, $\alpha = \alpha p$ and $\beta = \beta p$. By the pullback property there

is a unique morphism $w : D \rightarrow B'$ in \mathcal{C} such that $gw = \alpha$ and $d'w = \beta$. Then the morphism $wp : (D, p) \rightarrow H(B')$ in $\widehat{\mathcal{C}}$ satisfies the equalities $H(g)wp = \alpha$ and $H(d')wp = \beta$. If $w' : (D, p) \rightarrow H(B')$ is another morphism in $\widehat{\mathcal{C}}$ such that $H(g)w' = \alpha$ and $H(d')w' = \beta$, then $w' : D \rightarrow B'$ and $w' = w'p$ in \mathcal{C} . By the uniqueness of $(B \times_C C', g, d')$, it follows that $w' = w'p = wp$. Hence $(H(B) \times_{H(C)} H(C'), H(g), H(d'))$ is a pullback of $H(d)$ and $H(h)$ in $\widehat{\mathcal{C}}$.

(ii) Assume first that $d : B \rightarrow C$ is a semi-stable cokernel in \mathcal{C} . Then $H(d)$ is a cokernel in $\widehat{\mathcal{C}}$ by a dual of (i) for pushouts. Let $Z \in \text{Im}(H)$ and let $\gamma : Z \rightarrow H(C)$ be a morphism in $\widehat{\mathcal{C}}$. Then $Z = H(C')$ for some object C' of \mathcal{C} and $\gamma = H(h)$ for some morphism $h : C' \rightarrow C$ in \mathcal{C} . There is a pullback $(B \times_C C', g, d')$ of d and h , and $(H(B) \times_{H(C)} H(C'), H(g), H(d'))$ is also a pullback by (i). Since d' is a cokernel in \mathcal{C} , $H(d')$ is a cokernel in $\widehat{\mathcal{C}}$ by a dual of (i). Hence $H(d)$ is an $\text{Im}(H)$ -semi-stable cokernel in $\widehat{\mathcal{C}}$.

Now assume that $H(d)$ is an $\text{Im}(H)$ -semi-stable cokernel in $\widehat{\mathcal{C}}$. Then d is a cokernel in \mathcal{C} by a dual of (i) for pushouts. Let $h : C' \rightarrow C$ be a morphism in \mathcal{C} . There is a pullback $(H(B) \times_{H(C)} H(C'), \beta, v)$ of $H(d)$ and $H(h)$ in $\widehat{\mathcal{C}}$. By Lemma 3.1, $H(B) \times_{H(C)} H(C') \in \text{Im}(H)$. Then $H(B) \times_{H(C)} H(C') = H(B')$ for some object $B' \in \mathcal{C}$, $v = H(d')$ and $\beta = H(g)$ for some morphisms $d' : B' \rightarrow C'$ and $g : B' \rightarrow B$ in \mathcal{C} . By (i) it follows that $(B \times_C C', g, d')$ is a pullback of d and h in \mathcal{C} . Moreover, d' is a cokernel in \mathcal{C} , because v is a cokernel in $\widehat{\mathcal{C}}$. Hence d is a semi-stable cokernel in \mathcal{C} . \square

Now we may obtain the absolute version for an arbitrary additive category of the result on relative semi-stable cokernels given in Proposition 2.7 for a weakly idempotent complete additive category. As noted in the introduction, every idempotent complete category is weakly idempotent complete.

Proposition 3.3. *Let $d : B \rightarrow C$ and $p : C \rightarrow D$ be morphisms in \mathcal{C} such that p has a kernel and $pd : B \rightarrow D$ is a semi-stable cokernel. Then p is a semi-stable cokernel.*

Proof. Immediate by Propositions 2.7 and 3.2. \square

In fact, Proposition 3.3 and its dual show that the semi-stable kernels and the semi-stable cokernels satisfy Quillen's "obscure axiom".

4. THE MAXIMAL EXACT STRUCTURE

We consider the following concept of exact category given by Quillen [13], as simplified by Keller [9].

Definition 4.1. By an *exact category* we mean an additive category \mathcal{C} endowed with a distinguished class \mathcal{E} of short exact sequences satisfying the axioms [E0], [E1], [E2] and [E2^{op}] below. The short exact sequences in \mathcal{E} are called *conflations*, whereas the kernels and cokernels appearing in such exact sequences are called *inflations* and *deflations* respectively.

[E0] The identity morphism $1_0 : 0 \rightarrow 0$ is a deflation.

[E1] The composition of two deflations is again a deflation.

[E2] The pullback of a deflation along an arbitrary morphism exists and is again a deflation.

[E2^{op}] The pushout of an inflation along an arbitrary morphism exists and is again an inflation.

We should note that the duals of the axioms [E0] and [E1] on inflations as well as both sides of Quillen's "obscure axiom" hold in any such exact category [9]).

Having prepared the necessary properties on (relative) semi-stable kernels and cokernels in the previous sections, now we are in a position to give our main result, which generalizes [3, Theorem 3.5]. We omit the proof, because it follows the same path as the proof of the cited result, using Lemma 2.4, Propositions 2.5 and 2.6 for $\mathcal{H} = \mathcal{C}$, Proposition 3.3 and their duals.

Theorem 4.2. *Let \mathcal{C} be an additive category. Then the stable short exact sequences define a maximal exact structure on \mathcal{C} .*

We end with two applications to categories of chain complexes and categories of projective spectra.

Corollary 4.3. *Let \mathcal{C} be an additive category and let $\mathbf{Ch}(\mathcal{C})$ be the category of chain complexes over \mathcal{C} . Then the maximal exact structure on $\mathbf{Ch}(\mathcal{C})$ consists of the short exact sequences which are stable short exact sequences in \mathcal{C} in each degree.*

Proof. If the additive category \mathcal{C} has an exact structure \mathcal{E} , then the additive category $\mathbf{Ch}(\mathcal{C})$ also has an exact structure $\mathbf{Ch}(\mathcal{E})$ whose conflations are the short exact sequences which are conflations from \mathcal{E} in each degree (e.g., see [2, Lemma 9.1]). In particular, if $\mathcal{E}_{\max}^{\mathcal{C}}$ is the maximal exact structure on \mathcal{C} given by Theorem 4.2, then $\mathbf{Ch}(\mathcal{E}_{\max}^{\mathcal{C}})$ is an exact structure on $\mathbf{Ch}(\mathcal{C})$, included in the maximal exact structure $\mathcal{E}_{\max}^{\mathbf{Ch}(\mathcal{C})}$ on $\mathbf{Ch}(\mathcal{C})$.

Now let $A \xrightarrow{i} B \xrightarrow{d} C$ be a conflation from $\mathcal{E}_{\max}^{\mathbf{Ch}(\mathcal{C})}$, that is, a stable short exact sequence in $\mathbf{Ch}(\mathcal{C})$ by Theorem 4.2. Set a degree n , let X be an object of \mathcal{C} and let $\alpha : X \rightarrow C^n$ be a morphism in \mathcal{C} . Then X can be viewed as a chain complex C' concentrated in degree n , and we may define a chain map $h : C' \rightarrow C$ by $h^n = \alpha$ and $h^m = 0$ for every $m \neq n$. By Lemma 2.4, the pullback of d and h yields the following commutative diagram in $\mathbf{Ch}(\mathcal{C})$:

$$\begin{array}{ccccc} A & \xrightarrow{i'} & B' & \xrightarrow{d'} & C' \\ \parallel & & \downarrow g & & \downarrow h \\ A & \xrightarrow{i} & B & \xrightarrow{d} & C \end{array}$$

Then i' is a semi-stable kernel by the dual of Proposition 3.3, and d' is a semi-stable cokernel. Hence $A \xrightarrow{i'} B' \xrightarrow{d'} C'$ is a conflation from $\mathcal{E}_{\max}^{\mathbf{Ch}(\mathcal{C})}$. Then $A^n \xrightarrow{i'^n} B'^n \xrightarrow{d'^n} C'^n$ is a kernel-cokernel pair in \mathcal{C} , and so $A^n \xrightarrow{i^n} B^n \xrightarrow{d^n} C^n$ is a stable short exact sequence in \mathcal{C} . Thus $A \xrightarrow{i} B \xrightarrow{d} C$ is a conflation from $\mathbf{Ch}(\mathcal{E}_{\max}^{\mathcal{C}})$, which shows that $\mathbf{Ch}(\mathcal{E}_{\max}^{\mathcal{C}}) = \mathcal{E}_{\max}^{\mathbf{Ch}(\mathcal{C})}$. \square

Following [4, Definition 7.1] a *projective spectrum* $X = (X_n, X_m^n)$ with values in a category \mathcal{C} consists of a sequence $(X_n)_{n \in \mathbb{N}}$ of objects of \mathcal{C} and morphisms $X_m^n : X_m \rightarrow X_n$ in \mathcal{C} defined for $n \leq m$ such that $X_n^n = 1_{X_n}$ for every $n \in \mathbb{N}$, and $X_n^k \circ X_m^n = X_m^k$ for $k \leq n \leq m$. A *morphism of*

projective spectra $f : X \rightarrow Y$ between two projective spectra $X = (X_n, X_m^n)$ and $Y = (Y_n, Y_m^n)$ consists of a sequence $(f_n)_{n \in \mathbb{N}}$ of morphisms $f_n : X_n \rightarrow Y_n$ in \mathcal{C} such that $f_n \circ X_m^n = Y_m^n \circ f_m$ for $n \leq m$. If \mathcal{C} is additive, then so is the category of projective spectra with values in \mathcal{C} .

Corollary 4.4. *Let \mathcal{C} be an additive category and let $\mathbf{P}(\mathcal{C})$ be the category of projective spectra with values in \mathcal{C} . Then the maximal exact structure on $\mathbf{P}(\mathcal{C})$ consists of the short exact sequences $A \xrightarrow{i} B \xrightarrow{d} C$ for which $A_n \xrightarrow{i_n} B_n \xrightarrow{d_n} C_n$ are stable short exact sequences in \mathcal{C} for every $n \in \mathbb{N}$.*

Proof. It is similar to the proof of [4, Corollary 7.4], using Theorem 4.2. \square

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